JOURNAL OF FUNCTIONAL ANALYSIS 7, 311–322 (1971)

A Regularity Theorem for Parabolic Equations

C. BARDOS*

Département de Mathématiques, Faculté des Sciences de Paris

Communicated by J. L. Lions

Received April 28, 1970

We consider the solution in a Hilbert space H of a parabolic equation of the following type:

$$u'(t) + A(t)\,u(t) = 0; \qquad u(0) = u_0\,,$$

where $A(t)$ is an elliptic operator depending on t. We prove, under suitable hypotheses on $A(t)$, an abstract regularity theorem, generalising the usual result (see J. L. LIONS, "Équations Différentielles Opérationnelles et Problèmes aux Limites," Chapter 5.3, Springer-Verlag, Berlin/New York, 1961). We give an example of application.

In the proof of the main theorem we use the square root $A^{1/2}(t)$ of the operator $A(t)$ and we consider the Banach space $D(A^{1/2}(t))$.

INTRODUCTION

We consider the solution in a Hilbert space H of a parabolic evolution equation of the following type:

$$u'(t) + A(t)\,u(t) = f(t),$$
$$u(0) = u_0\,,$$

where $A(t)$ is an elliptic operator defined by a Banach space V which is included with continuous injection in H, dense in H, and by a sesquilinear form a $(t; u, v)$ is continuous and coercive on V. We prove under suitable hypotheses on $A(t)$ an abstract regularity theorem generalising the usual results about parabolic operators (see Lions, Ref. [8], Chapter 5.3, and also Agranovic and Visik, Ref. [1], and Lions and Magenes, Ref. [7]) and we show, by an example, how to deduce the regularity of the solution from the regularity of the operator $A(t)$.

* Chargé d'enseignement à la Faculté des Sciences de Paris.

1. The Abstract Theorem

Let V be a reflexive Banach space topologically included with continuous injection in a Hilbert space H; we denote by $(\ ,\ )$ and $|\cdot|$ the scalar product and the norm on H and by $\|\cdot\|$ the norm on V.

We suppose that V is dense in H and thus, identifying H with its antidual, we have the usual injections,

$$V \subset H \subset V^*. \tag{1.1}$$

Let $0 < T < +\infty$ and let $t \mapsto A(t)$ be a measurable mapping from $[0, T]$ into the space $\mathscr{L}(V, V^*)$ of bounded operators from V into V^* such that

$$|(A(t), u, v)| \leqslant M \| u \| \cdot \| v \| \tag{1.2}$$

and

$$\operatorname{Re}(A(t)\, u, u) \geqslant \alpha \| u \|^2 \tag{1.3}$$

for almost every t, M and α being independent of t.

We know (see Lions, Ref. [6]) that for every couple

$$(u_0, f) \in H \times L^2(0, T; V^*)$$

there exists a unique function $u \in \mathscr{C}(0, T; H) \cap L^2(0, T; V)$, which is a solution of the equation

$$\begin{aligned} u'(t) + A(t)\, u(t) &= f(t), \\ u(0) &= u_0 . \end{aligned} \tag{1.4}$$

For any $t \in [0, T]$ the operator $A(t)$ restricted to H defines an unbounded operator still denoted by $A(t)$, with domain

$$D(A(t)) = \{v \in V, A(t) \cdot v \in H\},$$

$A(t)$ is maximal positive ($-A(t)$ is the generator of a strongly continuous contraction semigroup). Thus $A(t)$ has a square root $A^{1/2}(t)$ with domain $D(A^{1/2}(t))$ (see Kato, Ref. [3], p. 282), given by the formula

$$A^{-1/2}(t) = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2}(A(t) + \lambda)^{-1}\, d\lambda. \tag{1.5}$$

Of course we have

$$D(A(t)) \subset D(A^{1/2}(t)) \subset H,$$

every space being dense in the following space.

From (1.3) it follows that the norms of $D(A(t))$ and $D(A^{1/2}(t))$ are given by

$$|v|_{D(A(t))} = |A(t) \cdot v|; \qquad |v|_{D(A^{1/2}(t))} = |A^{1/2}(t) \cdot v|; \tag{1.6}$$

$A^{1/2}(t)$ is also maximal positive and the semigroup $e^{-\xi A^{1/2}(t)}$ commutes with $A(t)$, i.e.,

$$e^{-\xi A^{1/2}(t)}(D(A(t)) \subset D(A(t)) \tag{1.7}$$

and

$$A(t)\, e^{-\xi A^{1/2}(t)} v = e^{-\xi A^{1/2}(t)} A(t) v, \qquad \forall v \in D(A(t)). \tag{1.8}$$

Hence for any $v \in D(A(t))$ we have

$$A^{1/2}(t) \left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} A(t) \cdot v = A(t)\, A^{1/2}(t) \left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} \cdot v \tag{1.9}$$

Finally we denote by $A^*(t)$ the adjoint of $A(t)$ in $\mathscr{L}(V, V^*)$; $A^*(t)$ restricted to H defines an unbounded operator which, of course, coincides with the adjoint of $A(t)$ considered as an unbounded operator in H.

THEOREM 1.1. *We assume that*

(i) $D(A^{1/2}(t)) = D(A^{*1/2}(t)) = V \qquad \forall t \in [0, T]$,

the equality being algebraic and topological; that is, there exist two positive constants independent of t, such that for any $u \in V$

$$C_1 \| u \| \leqslant | A^{1/2}(t) u | \leqslant C_2 \| u \|; \qquad C_1 \| u \| \leqslant | A^{*1/2}(t) u | \leqslant C_2 \| u \|; \tag{1.10}$$

(ii) $A^{1/2}(t)$ *considered as a bounded operator from* V *to* V^* *is differentiable, and we have*

$$|(A^{1/2}(t))'|_{\mathscr{L}(V,V^*)} \leqslant N.$$

Then if f *belongs to* $L^2(0, T; H)$ *and if* u_0 *belongs to* V, *the solution* u *of* (1.4) *belongs to* $L^2(0, T; D(A(t))$ (*i.e., for almost every* t, $u(t) \in D(A(t))$, $A(t)\, u(t) \in L^2(0, T; H)$), *and* $u'(t)$ *belongs to* $L^2(0, T; H)$.

Moreover if

(iii) $D(A(t))$ *is independent of* t

then the solution u *of* (1.4) *belongs to* $\mathscr{C}(0, T; V)$.

Remark. The hypothesis (i) is satisfied as soon as there exists a Banach space X, $X \subset H$ such that

(1) V is a closed subspace of $T(2, 0; X, H)$;

(2) $\forall t \in [0, T]$, $D(A(t)) \subset X$ and $D(A^*(t)) \subset X$, the inclusion being algebraic and topological.

Of course, (1) and (2) are trivially satisfied with $X = V$ when $D(A(t)) = D(A^*(t))$ (For notations and proofs see Lions, Ref. [5].) Proposition 1.1, which will be proved further on, gives sufficient conditions implying (ii).

Proof of Theorem 1.1. Since $D(A^{*1/2}(t)) = V$, the operator $A^{*1/2}(t)(I + (1/n)\, A^{*1/2}(t))^{-1}$, defines restricted to V, a bounded operator; therefore its adjoint $A^{1/2}(t)(I + (1/n)\, A^{1/2}(t))^{-1}$ extended to V^*, defines a bounded operator in V^*. Thus, we can deduce from (1.4) that

$$A^{1/2}(t)\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} u' + A^{1/2}(t)\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} A(t)u$$
$$= A^{1/2}(t)\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} f. \tag{1.11}$$

Since the operators

$$A^{1/2}(t)\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} A(t)$$

and

$$A(t)\, A^{1/2}(t)\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1}$$

are bounded from V to V^* and since $D(A(t))$ is dense in V, (1.9) remains true for any v in V.

On the other hand, from (ii) the operator

$$A^{1/2}(t)\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1} = nI - n\left(I + \frac{1}{n} A^{1/2}(t)\right)^{-1},$$

considered as an operator from V to V^*, is differentiable, and we have

$$\begin{aligned}\left(A^{1/2}(t)\left(I+\frac{1}{n}A(t)\right)^{-1}\right)' &= -n\left(\left(I+\frac{1}{n}A^{1/2}(t)\right)^{-1}\right)' \\ &= \left(I+\frac{1}{n}A^{1/2}(t)\right)^{-1}(A^{1/2}(t))'\left(I+\frac{1}{n}A^{1/2}(t)\right)^{-1}.\end{aligned} \tag{1.12}$$

From (ii) we note that this last term is bounded in $\mathscr{L}(V, V^*)$ by N.

Henceforth we will denote by $u_n(t)$ the function

$$u_n(t) = A^{1/2}(t)\left(I+\frac{1}{n}A^{1/2}(t)\right)^{-1}u(t).$$

From (1.11) and (1.12) we deduce that

$$\begin{aligned}u_n' + A(t)\,u_n(t) = A^{1/2}(t)\left(I+\frac{1}{n}A^{1/2}(t)\right)^{-1}f \\ - \left(A^{1/2}(t)\left(I+\frac{1}{n}A^{1/2}(t)\right)^{-1}\right)' u.\end{aligned} \tag{1.13}$$

Multiplying (1.13) by u_n and integrating from zero to t, we obtain

$$\begin{aligned}&\frac{1}{2}|u_n(t)|^2 + \alpha\int_0^t \|u_n(\sigma)\|^2\,d\sigma \\ &\leqslant \frac{1}{2}|u_n(0)|^2 + \int_0^t \left\|A^{1/2}(\sigma)\left(I+\frac{1}{n}A^{1/2}(\sigma)\right)^{-1}f(\sigma)\right\|\cdot\|u_n(\sigma)\|\,d\sigma \\ &\quad + \int_0^t \left\|\left(A^{1/2}(\sigma)\left(I+\frac{1}{n}A^{1/2}(\sigma)\right)^{-1}\right)' u(\sigma)\right\|\cdot\|u_n(\sigma)\|\,d\sigma.\end{aligned} \tag{1.14}$$

From (1.10) it follows that $A^{*1/2}(t)$ is bounded from V to H; therefore its adjoint $A^{1/2}(t)$ can be extended as a bounded operator from H to V^*, and its norm is bounded by C_2. Thus, since f belongs to $L^2(0, T; H)$, we have

$$\begin{aligned}&\int_0^t \|A^{1/2}(\sigma)(I+1/n\,A^{1/2}(\sigma))^{-1}f(\sigma)\|\cdot\|u_n(\sigma)\|\,d\sigma \\ &\leqslant c_2\int_0^t |(I+1/n\,A^{1/2}(\sigma))^{-1}f(\sigma)|\cdot\|u_n(\sigma)\|\,d\sigma\end{aligned} \tag{1.15}$$

$$\leqslant c_2\int_0^t |f(\sigma)|\cdot\|u_n(\sigma)\|\,d\sigma. \tag{1.16}$$

On the other hand, from (1.12) we obtain

$$\int_0^t \|(A^{1/2}(\sigma)(I + 1/n\, A^{1/2}(\sigma))^{-1})'\, u(\sigma)\| \cdot \| u_n(\sigma)\|\, d\sigma$$
$$\leqslant N \int_0^t \| u(\sigma)\| \, \| u_n(\sigma)\|\, d\sigma. \tag{1.16}$$

Since

$$\int_0^t \| u(\sigma)\|^2\, d\sigma \leqslant C \left\{ | u_0 |^2 + \int_0^t | f(\sigma)|^2 \right\} d\sigma \tag{1.17}$$

(we shall use C to denote several constants all independent of t and n), we may deduce from (1.14), (1.15), and (1.16) that

$$1/2\, | u_n(t)|^2 + \alpha/2 \int_0^t \| u_n(\sigma)\|^2\, d\sigma$$
$$\leqslant 1/2\, | u_n(0)|^2 + C \left\{ | u_0 |^2 + \int_0^t | f(\sigma)|^2\, d\sigma \right\}. \tag{1.18}$$

Since u_0 belongs to $V = D(A^{1/2}(t))$, when $n \to \infty$,

$$1/2\, | u_n(0)|^2 = 1/2\, |(I + 1/2\, A^{1/2}(0))^{-1}\, A^{1/2}(0)\, u_0 |^2$$

remains bounded.

On the other hand, $(I + 1/n\, A^{1/2}(t))^{-1}\, u(t)$ converges to $u(t)$ in $L^2(0, T; H)$, $A^{1/2}(t)(I + 1/n\, A^{1/2}(t))^{-1}\, u(t)$ belongs to $L^2(0, T; V)$, and, since $D(A^{1/2}(t)) = V$, we have

$$A(t)(I + 1/n\, A^{1/2}(t))^{-1}\, u(t) \in L^2(0, T; H)$$

and

$$\int_0^T | A(t)(I + 1/n\, A^{1/2}(t))^{-1}\, u(t)|^2\, dt$$
$$\leqslant C_2 \int_0^T \| A^{1/2}(t)(I + 1/n\, A^{1/2}(t)^{-1}\, u(t)\|^2\, dt. \tag{1.19}$$

(1.18) implies that the right side of (1.20) is bounded when $n \to \infty$; thus $(I + 1/n\, A^{1/2}(t))^{-1}\, u(t)$ converges to $u(t)$ weakly in $L^2(0, T; D(A(t)))$. Thus we have proved that $u(t)$ belongs to $L^2(0, T; D(A(t))$. Equation (1.4) shows that u' belongs to $L^2(0, T; H)$. Finally, since

$$V = (D(A^{1/2}(t)) = [D(A(t)), H]_{1/2}\,,^1 \tag{1.20}$$

[1] See Lions, Ref. [5].

we deduce from

$$u(t) \in L^2(0, T; D(A(t))),$$
$$u'(t) \in L^2(0, T; H), \tag{1.21}$$

that if $D(A(t))$ is independent of t, then

$$u(t) \in \mathscr{C}(0, T; V). \tag{1.22}$$

PROPOSITION 1.1. *Suppose*

(1) *The hypothesis* (i) *is satisfied;*

(2) *The mapping* $t \mapsto A(t)$ *from* $[0, T]$ *to* $\mathscr{L}(V, V^*)$ *is differentiable;*

(3) *There exists some constant* Q *(independent of* $t \in [0, T]$*) such that*

$$|(A(t))|_{\mathscr{L}(V,V^*)} \leqslant Q. \tag{1.23}$$

Then the hypothesis (ii) *is satisfied.*

Proof. For any $v \in D(A(t))$ we have (see Kato, Ref. [3])

$$A^{1/2}(t) \cdot v = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2}(\lambda + A(t))^{-1} A(t) \cdot v \, d\lambda, \tag{1.24}$$

and (1.24) remains true for any v in V because $D(A(t))$ is dense in V. By a computation similar to (1.12) we have

$$((\lambda + A(t))^{-1} A(t))' = -(\lambda + A(t))^{-1}(A(t))'(\lambda + A(t))^{-1}; \tag{1.25}$$

the right side of (1.25) is bounded in $\mathscr{L}(V, V^*)$ uniformly in t by $C < +\infty$ for $\lambda \to 0$ and by $C\lambda^{-2}$ for $\lambda \to \infty$. Thus (1.24) is differentiable and we have

$$(A(t)\, v)' = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2}(\lambda + A(t))^{-1}(A(t))'(\lambda + A(t)^{-1} \cdot v \, d\lambda \tag{1.26}$$

Finally, it follows from (1.23) that the right side of (1.26) is uniformly bounded in t.

For any integer $k \geqslant 0$, we denote by $A^k(t)$ the operator defined inductively by the relations $A^0(t) = I$, $A^1(t) = A(t)$, and

$$D(A^k(t)) = \{v;\, v \in D(A^{k-1}(t)) \text{ and } A^{k-1}(t) \cdot v \in D(A(t))\},$$

where $A^k(t) \cdot v = A(t)(A^{k-1}(t) \cdot v)$.

Let $D(A^{k+1/2}(t)) = \{v \in D(A^k(t)) \mid A^k(t) \in D(A^{1/2}(t))\}$, and denote by $L^2(0, T; D(A^k(t)))$ the space

$$L^2(0, T; D(A^k(t))) = \Big\{v(t) \in L^2(0, T; H); v(t) \in D(A^k(t)) \text{ for almost every } t; \int_0^T | A^k(t)\, v(t)|^2\, dt < +\infty\Big\}.$$

Then $L^2(0, T; D(A^k(t))))$ is a Hilbert space for the natural scalar product.

THEOREM 1.2. *We assume that the hypothesis* (i) *of Theorem* 1.1 *and the hypothesis* (1.23) *of Proposition* 1.1 *are satisfied and, moreover, that* $A'(t)$ *defines by restriction to* $D(A^1(t))$ $(1 \leqslant 1 \leqslant k)$ *a linear bounded operator from*

$$L^2(0, T; D(A^l(t)) \quad \text{to} \quad L^2(0, T; D(A^{l-1}(t))).$$

If

$$(u_0, f) \in D(A^{k+1/2}(0)) \times L^2(0, T; D(A^k(t))),$$

then $u(t)$ *belongs to* $L^2(0, T; D(A^{k+1}(t)))$ *and* $u'(t)$ *belongs to* $L^2(0, T; D(A^k(t)))$. *In addition, if the spaces* $D(A^k(t))$ *and* $D(A^{k+1}(t))$ *are independent of t, then* $u(t)$ *belongs to the space*

$$\mathscr{C}(0, T; [D(A^{k+1}(t)), D(A^k(t))]_{1/2}).$$

Proof. By induction: From Theorem 1.1, we deduce that Theorem 1.2 is true for $k = 0$; if we assume that the theorem is true for $k - 1$ $(1 \leqslant k)$ when f belongs to $L^2(0, T; D(A^k(t)))$ and u_0 to $D(A^{k+1/2}(t))$, then the solution $u(t)$ of (1.4) belongs to $L^2(0, T; D(A^k(t)))$. Put $v(t) = A(t)\, u(t)$; since $A'(t)$ is bounded from $L^2(0, T; D(A^k(t)))$ to $L^2(0, T; D(A^{k-1}(t)))$, $v(t)$ is the solution of the problem

$$\begin{aligned} v'(t) + A(t)\, v(t) &= A(t) f(t) + A'(t)\, u(t), \\ v(0) &= A(0)\, u_0 . \end{aligned} \tag{1.27}$$

Since $u_0 \in D(A^{k+1/2}(0))$, $A(0)\, u_0 \in D(A^{k-1/2}(0))$, and since $u(t)$ belongs to $L^2(0, T; D(A^k(t)))$ we have

$$A(t) f(t) + A'(t)\, u(t) \in L^2(0, T; D(A^{k-1}(t))). \tag{1.28}$$

So using once more the induction hypothesis we deduce that $v(t)$ belongs to $L^2(0, T; D(A^k(t)))$; that is, $u(t) \in L^2(0, T; D(A^{k+1}(t)))$. The rest of the proof is as in Theorem 1.1.

We shall give now some results on the regularity in t of the solution of (1.4).

THEOREM 1.3. *We assume that the hypothesis* (i) *of Theorem* 1.1 *and the hypothesis* (1.23) *of Proposition* 1.1 *are satisfied, and moreover, that up to order p the successive derivatives $A^{(r)}(t)$ are linear operators from V to V^* uniformly bounded in t, whose restrictions to $D(A^l(t))$ define linear operators from $D(A^l(t))$ to $D(A^{l-1}(t))$ uniformly bounded in t. If*

$$f \text{ and its } p \text{ derivatives } f^{(1)}, f^{(2)}, \dots, f^{(p)} \text{ belong to } L^2(0, T; D(A^k(t)), \tag{1.29}$$

$$u_0 \in D(A^{k+1/2}(t)), \tag{1.30}$$

and if the sequence δ_1 defined by

$$\begin{cases} \delta_0 = u_0 , \\ \delta_l = f^{(l-1)} - \sum_{m=0}^{l-1} \binom{l-1}{m} A^{(m)}(0)\, \delta_{(l-1-m)} \end{cases} \tag{1.31}$$

satisfies

$$\delta_l \in D(A^{k+1/2}(0)), \tag{1.32}$$

then u and its p successive derivatives belong to $L^2(0, T; D(A^{k+1}(t)))$, and $u^{(p+1)}$ belongs to $L^2(0, T; D(A^k(t)))$. Moreover, if the spaces $D(A^k(t))$ and $D(A^{k+1}(t)))$ are independent of t we have, for every r, $0 \leqslant r \leqslant p$,

$$u^{(r)}(t) \in \mathscr{C}(0, T; [D(A^{k+1}(t)), D(A^k(t))]_{1/2}).$$

Proof. First we assume that $0 < k$ and prove the theorem by induction; in fact, if it is true for order $p - 1$, $u^{(p-1)}$ belongs to $L^2(0, T; V)$ and is a solution of an equation of the type

$$\begin{aligned} &(u^{(p-1)}(t))' + A(t)\, u^{(p-1)}(t) \\ &\qquad = f^{(p-1)} - \sum_{m=1}^{p-1} \binom{p-1}{m} A^{(m)}(t)\, u^{(p-1-m)}(t), \\ &u^{(p-1)}(0) = f^{(p-2)}(0) - \sum_{m=0}^{p-2} \binom{p-2}{m} A^{(m)}(0)\, u^{(p-2-m)}(0) \\ &\qquad = f^{(p-2)}(0) - \sum_{m=0}^{p-2} \binom{p-2}{m} A^{(m)}(0)\, \delta_{(p-2-m)} = \delta_{(p-1)} .^{2} \end{aligned} \tag{1.33}$$

[2] Under the induction hypothesis we have $u^{(m)}(t) \in \mathscr{C}(0, T; V)$, for any m, $0 \leqslant m < p - 1$, which gives sense to $A^{(m)}(0)\, u^{(p-2-m)}(0)$ in (1.33).

From the induction hypothesis we deduce that

$$u^{(p-1)}(t) \in L^2(0, T; D(A^{k+1}(t))) \tag{1.34}$$

and that

$$u^{(p)}(t) \in L^2(0, T; D(A^k(t))).$$

Therefore we can derive the first equation of (1.33) and we obtain

$$(u^{(p)}(t))' + A(t)\, u^{(p)}(t) = f^{(p)}(t) - \sum_{m=1}^{p} \binom{p}{m} A^{(m)}(t)\, u^{(p-m)}(t),\ u^{(p)}(0) = \delta_p . \tag{1.35}$$

Since, under the induction hypothesis, the right side of the first equation of (1.35) belongs to $L^2(0, T; D(A^k(t)))$, we deduce from Theorem 1.2 that

$$u^{(p)}(t) \in L^2(0, T; D(A^{k+1}(t))); u^{(p+1)}(t) \in L^2(0, T; D(A^k(t))). \tag{1.36}$$

If $k = 0$ we introduce sequences f_ϵ and $u_0{}^\epsilon$ such that

$$f_\epsilon{}^l \in L^2(0, T; V), \qquad 1 \leqslant l \leqslant p, \tag{1.37}$$

$$\delta_l{}^\epsilon \in D(A^{3/2}(t)), \qquad 1 \leqslant l \leqslant p \tag{1.38}$$

($\delta_l{}^\epsilon$ being defined from $u_0{}^\epsilon$ by (1.31)), and such that $f_\epsilon^{(1)}$ converges to $f^{(1)}$ in $L^2(0, T; H)$ while $\delta_l{}^\epsilon$ converges to δ_l in V.

Using the Theorem 1.3 we can show that $u_\epsilon(t)$ is the solution of

$$u_\epsilon' + A(t)\, u_\epsilon = f_\epsilon ,$$
$$u_\epsilon(0) = u_0{}^\epsilon,$$

and its p derivatives belong to $L^2(0, T; D(A(t)))$; moreover it is easy to see that they are bounded in $L^2(0, T; D(A(t)))$ independently of ϵ. Thus, we can deduce Theorem 1.3, when $k = 0$, by passing to the limit.

2. An Application

We shall study the regularity of the solution of a problem of evolution related to a degenerate operator introduced by Baouendi and Goulaouic [2].

Let Ω be a bounded open set in R^n; assume that $\delta\Omega$, the boundary of Ω, is a manifold of dimension $n - 1$ and of class C^∞. Let φ be a C^∞ mapping from R^n to R such that

$$\Omega = \{x \in R^n, \varphi(x) > 0\},$$
$$\delta\Omega = \{x \in R^n, \varphi(x) = 0\},$$
$$\delta\varphi(x) \neq 0 \quad \text{if} \quad x \in \delta\Omega.$$

We denote by V the space of distributions $u \in \mathscr{D}(\Omega)$ such that $\varphi \cdot u \in L^2(\Omega)$ and $\varphi \cdot D^i u \in L^2(\Omega)$ $(\forall i, 1 \leqslant i \leqslant n)$ with the natural scalar product.

On the other hand, for any $t \in [0, T]$, let $A(t)$ be the linear operator from V to V' defined by the sesquilinear form

$$a(t; u, v) = \sum_{i,j} \int_\Omega a_{i,j}(x, t)\, \varphi(x)\, D^i u \cdot D^j v \, dx + \int_\Omega a_0(x, t)\, u \cdot v \, dx,$$

where $a_0(x, t)$ and $a_{i,j}(x, t)$ belong to $C^\infty[\Omega \times 0, T]$ and satisfy the hypothesis

$$\operatorname{Re} \sum_{i,j} a_{i,j}(x, t)\, \xi_i \bar{\xi}_j \geqslant \alpha \sum_i |\xi_i|^2 \qquad \forall \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{C}^n,$$

where $\alpha > 0$ is independent of x, t, and ξ.
Under these hypotheses we know (see Ref. [2]) that we have

$$D(A^k(t)) = \{u \in L^2(\Omega), \varphi^l \cdot u \in H^{k+1}(\Omega) \qquad \forall l,\ 0 \leqslant l \leqslant k.^3\} \tag{2.1}$$

In fact, (2.1) implies that the spaces $D(A(t))$ and $D(A^*(t))$ are contained in the space of functions u such that

$$\{u \in H^1(\Omega), \varphi \cdot u \in H^2(\Omega)\}.$$

Hence we can deduce from Theorem 6.1 of Lions Ref. [5], that

$$D(A^{1/2}(t)) = D(A^{*1/2}(t)) = V,$$

and, then using Theorems 1.1, 1.2, and 1.3, we obtain

THEOREM 2.1. *If, on one hand,* $\forall i$, $0 \leqslant i \leqslant p$,

$$\varphi^r(x) \cdot f(x, t) \in L^2(0, T; H^{k+r}(\Omega)) \qquad (0 \leqslant r \leqslant k),$$

[3] Of course we denote by $H^{k+1}(\Omega)$ the usual Sobolev space (see Lions, Ref. [4]).

and if, on the other hand, $u_0 \in L^2(\Omega)$ *is such that the sequence* δ_l *defined by*

$$\delta_0 = u_0 ,$$

$$\delta_l = f^{(l-1)}(x, 0) + \sum_{m=0}^{l-1} \binom{l-1}{m} D^i \cdot D_t^m a_{i,j}(x, 0) \cdot \varphi(x) D^j \delta_{l-1-m} \quad (2.2)$$

$$- D_t^m a_0(x, 0)\, \varphi(x)\, \delta_{1-l-m} ,$$

satisfies the relation

$$\varphi^r(x)\, \delta_l(x) \in H^{k+1/2+r}(\Omega) \qquad \forall r,\ 0 \leqslant r \leqslant k, \quad (2.3)$$

then the solution $u(x, t)$ *of the problem*

$$D_t u - \sum D^i a_{i,j}(x, t) \cdot \varphi(x)\, D^j u + a_0(x, t) \cdot \varphi(x) \cdot u = f,$$

$$u(x, 0) = u_0 ,$$

satisfies the relations

$$\varphi^r(x)\, D_t^s u(x, t) \in L^2(0, T; H^{k+r}(\Omega)), \qquad 0 \leqslant s \leqslant p,\ 0 \leqslant r \leqslant k$$

and

$$\varphi^r(x)\, D_t^{p+1} u(x, t) \in L^2(0, T; H^{k+r}(\Omega)), \qquad 0 \leqslant r \leqslant k.$$

References

1. M. S. Agranovic and M. I. Visik, Elliptic problems with parameter and parabolics problems of general type, *Uspehi Mat. Nauk* **19**, No. 3 (1964), 53–161 (*Russian Math. Surveys* **19**, No. 3 (1964), 53–157).
2. M. S. Baouendi and C. Goulaouic, Régularité et théorie spectrale pour une classe d'opérateurs dégénérés, to appear.
3. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York/Berlin, 1966.
4. J. L. Lions, "Problèmes aux Limites dans les Équations aux Dérivées Partielles," Université de Montreal, Montreal, Canada, 1962.
5. J. L. Lions, Espaces d'interpolation et domaines de puissance fractionnaires d'opérateurs, *J. Math. Soc. Japan* **14**, No. 2 (1962).
6. J. L. Lions, "Équations Différentielles Opérationnelles dans les Espaces de Hilbert," edizioni Cremonese Roma, C.I.M.E. (1963).
7. J. L. Lions and E. Magenes, "Problèmes aux Limites Non Homogènes et Application," Dunod, Paris.
8. J. L. Lions, "Équations Différentielles Opérationnelles et Problèmes aux Limites," Springer-Verlag, New York/Berlin, 1961.